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Mixed-Mode Propulsion: Optimum Burn Profile for Two-Mode Systems

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Nomenclature

C = constant, $\rho_2(u_2 - u_1)/(\rho_1 - \rho_2)$
m = mass
u = exhaust velocity
$\bar{u}$ = average u for both modes operating concurrently
V = volume of propellant burned
v = vehicle velocity
ρ = propellant bulk density
$\bar{\rho}$ = average ρ for both modes operating concurrently

Subscripts

0, 1, 2, f = initial, mode-1, mode-2, final

Introduction

IT has been pointed out[1] that combining different propulsion modes in the same stage promises improved vehicle performance, cost and operating characteristics, and may make a reusable one-stage-to-orbit machine feasible. Reference 1 assumes a purely sequential burn for the two-mode system. Under this assumption, the analysis indicates that the higher density mode should be operated first. The question is left open, however, whether sequential or some form of overlapping burn profile constitutes the true optimum for maximizing ideal Δv. The purpose of this analysis is to answer that question.

Analysis

Generalization of the ideal rocket equation

Consider a mixed-mode rocket stage having a given initial mass m_0, in which the modes can be operated concurrently at arbitrarily varying flow rates. Thus,

$$\bar{\rho} = \bar{\rho}(V) \tag{1}$$

$$\bar{u} = \bar{u}(\bar{\rho}(V)) \tag{2}$$

Equation (1) represents the burn profile, and Eq. (2) represents the specific functional dependency of $\bar{u}$ upon $\bar{\rho}$.

Since the purpose of this analysis is to present as simply as possible an initial derivation of the optimum burn profile, consideration of gravity and drag forces and more sophisticated formulations of the rocket equation[2] will not be included here. For point-mass analysis, conservation of momentum gives

$$\bar{u}\bar{\rho}dV = \left[m_0 - \int_0^V \bar{\rho}(\tilde{V})d\tilde{V}\right]dv \tag{3}$$

Substituting Eqs. (1) and (2) and integrating

$$\Delta v = \int_0^{V_f} \frac{\bar{\rho}(V)\bar{u}(\bar{\rho}(V))dV}{m_0 - \int_0^V \bar{\rho}(\tilde{V})d\tilde{V}} \tag{4}$$

which is a generalized form of the ideal rocket equation where $\bar{\rho}$ and $\bar{u}$ are no longer constant as in the more familiar classical single-mode case.

Optimization for the two-mode case

To maximize Δv in Eq. (4), it is first necessary to specify the function $\bar{u}[\bar{\rho}(V)]$. This is done by writing the expressions for $\bar{\rho}$ and $\bar{u}$ for the two-mode case,

$$\bar{\rho} = (\dot{m}_1 + \dot{m}_2)/(\dot{m}_1/\rho_1 + \dot{m}_2/\rho_2) \tag{5}$$

$$\bar{u} = (\dot{m}_1 u_2 + \dot{m}_2 u_2)/(\dot{m}_1 + \dot{m}_2) \tag{6}$$

and solving simultaneously to obtain

$$\bar{u} = u_1 + C[(\rho_1/\bar{\rho}) - 1] \tag{7}$$

Now, if we let

$$m = m_0 - \int_0^V \bar{\rho}(V)dV \tag{8}$$

and substitute Eqs. (7) and (8) in Eq. (4), then

$$\Delta v = (C - u_1)\int_{m_0}^{m_f} \frac{dm}{m} + \rho_1 C \int_0^{V_f} \frac{dV}{m} \tag{9}$$

and this becomes upon integration of the first term,

$$\Delta v = (C - u_1)\log_e(m_f/m_0) + \rho_1 C \int_0^{V_f} \frac{dV}{m} \tag{10}$$

which is the generalized ideal rocket equation for the two-mode case.

If the optimum burn profile is continuous, then it should be possible to determine it by applying the calculus of variations to Eq. (10), which can be written

$$\Delta v = \text{constant} + \rho_1 C \int_0^{V_f} F(V, m)dV \tag{10a}$$

Application of calculus of variations is possible only where all variables are continuous and possess continuous first and second derivatives. To define extremals, it is necessary to satisfy the Euler-Lagrange equation which in this case reduces to

$$-\partial F/\partial m = 0 \tag{11}$$

Substituting Eq. (10a) in Eq. (11) and differentiating,

$$-\rho_1 C/m^2 = 0 \tag{12}$$

which requires $m^2 \to \infty$. This is a contradiction which implies that there exists no continuous solution within the limits ρ_1 and ρ_2. Thus, if an optimum solution exists (and Ref. 1 proves that it can), then that solution must involve a discontinuous burn profile.

Applying a more general approach to maximize Δv in Eq. (10), m in the second term of Eq. (10) must be taken as small as possible and still satisfy the boundary conditions of m_0 and m_f. As shown in Fig. 1, the m history must lie between the limits of the straight line curves whose slopes are $-\rho_1$ and $-\rho_2$. Regardless of what m_f value is selected it becomes apparent that the smallest m history can be attained only by burning first the higher density (ρ_1) mode and then switching to the lower density (ρ_2) burn only when the lower density slope ($-\rho_2$) will lead to m_f. This is true regardless of the value of m_f so long as m_f lies between the above bounds, as it must.

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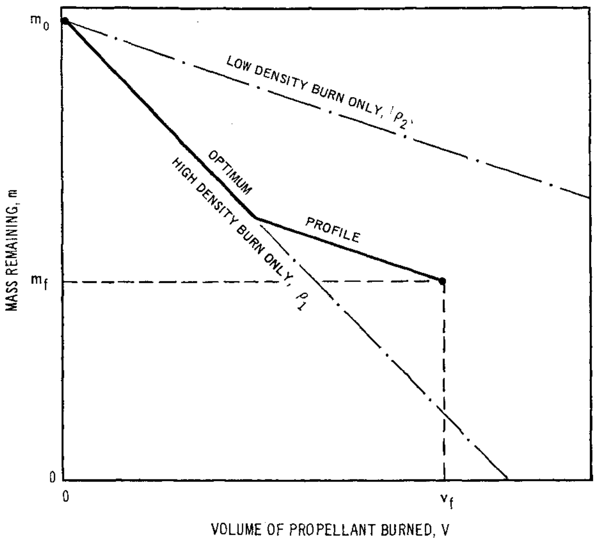

Fig. 1 Mass remaining vs volume of propellant burned.

It has been noted by Leitmann[3] that the application of optimal control theory to this problem will also yield an optimum solution. Results indicate that the optimum burn profile is sequential with the higher density mode preceding. It should be stressed, however, that the control found in this manner is extremal only and its optimality can be established by invoking existence or sufficiency theorems, or by recourse to direct reasoning as in the preceding paragraph.

Conclusion

To maximize ideal Δv for a two-mode one-stage rocket, the optimum burn profile is purely sequential, with the higher density mode operating first.

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Effects of Heat Addition in Divergent Nozzles with Application to MPD Thrusters

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A WIDE variety of MPD thruster configurations have been described in the literature, many of which have a recognizable area minimum or geometrical throat. Electrode arrangement varies greatly among the various designs, so that the bulk of the (usually large) electromagnetic energy input to the propellant gas can occur either upstream or downstream of the geometric throat. A large part of the total energy input is in the form of ohmic heating which influences the gas-dynamic contribution to the total thrust. The purpose of this Note is to point out that heat addition, when applied downstream of the geometric throat, can lead to unusual gas-dynamic behavior including the presence of shocks. In comparison, heating upstream of the throat produces no unexpected features.

These findings follow from the steady, one-dimensional gas-dynamic equations neglecting friction but taking area change and heat addition into account. The energy equation is represented by a simple specification of total temperature along the nozzle axis. Assumption of thermodynamic equilibrium, the perfect gas relation $p = \rho RT$, and the definition of total temperature in terms of Mach number results in a closed set of equations for ρ, p, T, T_0 and v.

The equation governing $M(x)$ can be written[1] as a function of distance along the axis (as measured from the throat):

$$\frac{dM^2}{dx} = 2M^2\left(1 + \frac{\gamma - 1}{2} M^2\right) \times \frac{\frac{1}{2}(1 + \gamma M^2)(1/T_0)dT_0/dx - (1/A)dA/dx}{1 - M^2} \quad (1)$$

where $T_0(x)$ and $A(x)$ are assumed to be known functions. Note that the actual independent variable is the area A since x could be cancelled out. (The scale of the x variable is consequently arbitrary, but it will be retained for convenience.)

Equation (1) may have more singular points than the one familiar from the classic adiabatic nozzle case. They occur if $M = 1$ and the following condition is satisfied:

$$(1/A)\, dA/dx = [(1 + \gamma)/2]\, (1/T_0)\, dT_0/dx \quad (2)$$

For demonstration purposes, we consider the example of an axially symmetric nozzle with a hyperbolic contour

$$R/R_* = [1 + \{(x/R_*) \tan \theta\}^2]^{1/2} \quad (3)$$

where θ is the asymptotic half-cone angle for the nozzle and R_* is the throat radius. The total temperature is described arbitrarily by:

$$\frac{T_0(x)}{T_{01}} = 1 + \frac{1}{2}(\tau - 1)\left\{1 + \tanh\left[\left(\frac{x - \bar{x}}{R_*}\right)\left(\frac{R_*}{\lambda}\right)\right]\right\} \quad (4)$$

where $\bar{x}$ is the location of the maximum in heating rate dT_0/dx, and λ is the characteristic axial distance of the heating zone. $\tau \equiv T_{02}/T_{01}$ is the ratio of total temperatures across the zone.

Closer examination shows that, in this example, Eq. (2) may be satisfied at any number of points from none to three depending on the value of the total temperature ratio. The absence of singularities corresponds to fully subsonic flow. One singular point will occur if τ is either less than 2.2 or more than 18 (approximately). In the first case the singular point occurs very near the throat, in the second case it is located a short distance downstream of the heating zone. Three singular points will occur if $2.2 \lesssim \tau \lesssim 18$. Two singular points represent intermediate, degenerate situations corresponding to either $\tau \simeq 2.2$ or $\tau \simeq 18$.

Figure 1 shows the nozzle contour defined by Eq. (3) and the total temperature and heating rate distributions defined by Eq. (4). The curves shown correspond to $\theta = 45°$, $\bar{x}/R_* = 2$, $R_*/\lambda = 2$ and $\tau = 4$. Figure 2 shows a family of $M(x)$ solutions for argon ($\gamma = \frac{5}{3}$) with τ as the parameter, obtained through the numerical integration of Eq. (1). In each curve, the integration was started at the appropriate singular point, where the derivative was evaluated analytically. Depending on the value of τ, the solutions may be grouped into three qualitatively different types.

The first type corresponds to $1.0 \leq \tau \lesssim 2.2$ and one singular

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